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Hardy–Rellich inequalities in domains of the Euclidean space

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A R T I C L E I N F O

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A B S T R A C T

For test functions supported in a domain of the Euclidean space we consider the Hardy–Rellich inequality: $\int |\Delta f|^2\,dx \ge C_2 \int |f|^2\delta^{-4}(x)\,dx$, where $C_2 = const \ge 0$ and $\delta(x)$ is the distance from x to the boundary of the domain. M.P. Owen proved that this inequality is valid in any convex domain with $C_2 = 9/16$ (M.P. Owen (1999) [21]). We examine the inequality in non-convex domains. It is proved that a positive constant C_2 for a plane domain exists if and only if its boundary is a uniformly perfect set. For a domain of dimension $d \ge 2$ we prove that the inequality holds with the sharp constant $C_2 = 9/16$, if the domain satisfies an exterior sphere condition with certain restriction on the radius of the sphere. In addition, we obtain similar results for the inequality $\int \delta^2(x)|\Delta f|^2\,dx \ge C_2^* \int |f|^2\delta^{-2}(x)\,dx$.

1. Introduction

In 1954 F. Rellich (see [22]) proved the following inequality

$$\int_{\mathbb{R}^d} |\Delta f|^2\,dx \ge 2^{-4}d^2(d-4)^2 \int_{\mathbb{R}^d} |f|^2|x|^{-4}dx \qquad \forall f \in C_0^\infty(\mathbb{R}^d \setminus \{0\}) \tag{1}$$

in the Euclidean space $\mathbb{R}^d$ for $d \ge 1$, $d \ne 2$, where Δf is the Laplacian of the test function $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$. In addition, Rellich showed that (1) is not valid for $d = 2$, even if one replaces the corresponding constant 1 by an arbitrarily small constant $\varepsilon > 0$. There are many papers by U.W. Schmincke, W. Allegretto, D.M. Bennett, E.B. Davies and A.M. Hinz, E. Mitidieri and other mathematicians (see, for instance, [23,1,13,16,20,14] and the literature therein), where one can find direct generalizations of (1).

Much less is known about inequalities of type (1) in domains $\Omega \subset \mathbb{R}^d$ with weight functions depending on other geometric quantities, different from $|x|$. In this paper we will study the case when weight functions depend on the distance function $\operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|,\ x \in \Omega$. More precisely, we are concerned with the following analogue of (1) due to M.P. Owen [21] (see, also, [12,11,17] for the same inequality with

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